

A NORMALITY CRITERION GENERALIZING GU'S RESULT

KULDEEP SINGH CHARAK AND VIRENDER SINGH

ABSTRACT. In this paper we prove a normality criterion for the families of meromorphic functions involving sharing of functions. Our result generalizes some of the earlier results on Gu's normality criterion.

1. Introduction and Main Results

It is assumed that the reader is familiar with the standard notions used in the Nevanlinna value distribution theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $S(r, f)$ etc., one may refer to [6].

A family \mathcal{F} of meromorphic functions defined on a domain $D \subseteq \overline{\mathbb{C}}$ is said to be normal in D if every sequence of elements of \mathcal{F} contains a subsequence which converges locally uniformly in D with respect to the spherical metric, to a meromorphic function or ∞ (see [10]).

Two nonconstant meromorphic functions f and g defined on D are said to share a meromorphic function ψ in D if $\overline{E}_f(\psi) = \overline{E}_g(\psi)$, where

$$\overline{E}_f(\psi) = \{z \in D : f(z) = \psi(z)\}.$$

The following Picard type theorem is one of the main result from Hayman's seminal paper [7]:

Theorem 1.1. (*Hayman's alternative*) *Let f be a nonconstant meromorphic function in \mathbb{C} , k a natural number and c a nonzero complex number. Then f or $f^{(k)} - c$ has a zero in \mathbb{C} . If f is transcendental, f and $f^{(k)} - c$ has infinitely many zeros in \mathbb{C} .*

In 1979, Y.X. Gu [5] proved the following normality criterion corresponding to Hayman's alternative:

Theorem 1.2. (*Gu's normality criterion*) *Let \mathcal{F} be a family of meromorphic functions defined in a domain D , and let k be a positive integer. If, for every function $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq 1$ in D , then \mathcal{F} is normal in D .*

Since then many variations of Theorem 1.2 have been obtained, for instance one can see [4, 8, 9, 11, 14, 15]. In fact Schwick [12] proved a more general version of Gu's result:

Theorem 1.3. *Let $\psi \neq 0$ be a meromorphic function in a domain D and $k \in \mathbb{N}$. Let \mathcal{F} be a family of meromorphic functions in D , such that $f \neq 0$ and $f^{(k)} \neq \psi$, and f and ψ have no common poles for each $f \in \mathcal{F}$. Then \mathcal{F} is normal in D .*

2010 *Mathematics Subject Classification.* 30D35, 30D45.

Keywords and phrases. Normal families, Meromorphic function, Shared function, Locally uniformly discrete sets.

The research work of the second author is supported by the CSIR India.

In 2004, Fang and Zalcman [3] proved the following generalization of Theorem 1.2 by considering the sharing of values:

Theorem 1.4. *Let k be a positive integer and b be a nonzero complex constant. Let \mathcal{F} be a family of meromorphic functions on D , all of whose zeros have multiplicity at least $k + 2$, such that for each pair of functions f and g in \mathcal{F} , f and g share the value 0, and $f^{(k)}$ and $g^{(k)}$ share the value b in D , then \mathcal{F} is normal in D .*

Recently, J. Chang [1] proved the following result by replacing the constant b by a holomorphic function:

Theorem 1.5. *Let $k \in \mathbb{N}$ and $h (\not\equiv 0)$ be a function holomorphic on D . Let \mathcal{F} be a family of meromorphic functions in D , all of whose zeros have multiplicity at least $k + 2$, such that for each pair of functions f and g in \mathcal{F} , f and g share the value 0, and $f^{(k)}$ and $g^{(k)}$ share the function h . Suppose additionally that at each common zero of f and h for every $f \in \mathcal{F}$, the multiplicities m_f for f and m_h for h satisfy $m_f \geq m_h + k + 1$ for $k > 1$ and $m_f \geq 2m_h + 3$ for $k = 1$. Then, \mathcal{F} is normal in D .*

Examples are also given in [1] for the sharpness of conditions in Theorem 1.5. Working in this direction, we prove the following generalization of Theorem 1.5 :

Theorem 1.6. *Let \mathcal{F} be a family of meromorphic functions in a domain D , and let k be a positive integer. Suppose that ϕ is a holomorphic function on D and ψ is a meromorphic function on D such that $\phi^{(k)}(z) \not\equiv \psi(z)$. Suppose that for each pair of functions f and g in \mathcal{F} , f and g share ϕ , and $f^{(k)}$ and $g^{(k)}$ share the function ψ . Suppose further that*

- (1) *every $f \in \mathcal{F}$, $f - \phi$ has zeros of multiplicity at least $k + 2$,*
- (2) *for every common zero of $f - \phi$ and $\psi - \phi^{(k)}$, the multiplicities $m_{f-\phi}$ for $f - \phi$ and $m_{\psi-\phi^{(k)}}$ for $\psi - \phi^{(k)}$ satisfy*

$$m_{f-\phi} \geq m_{\psi-\phi^{(k)}} + k + 1 \text{ for } k > 1, \text{ and}$$

$$m_{f-\phi} \geq 2m_{\psi-\phi^{(k)}} + 3 \text{ for } k = 1,$$

- (3) *for every $f \in \mathcal{F}$, f and ψ have no common poles in D .*

Then \mathcal{F} is normal on D .

Remark 1.7. If $\phi \equiv 0$ and ψ is a holomorphic function, then Theorem 1.6 reduces to Theorem 1.5. Thus, the conditions (1) and (2) in Theorem 1.6 can easily be seen to be essential.

Example 1.8. Consider the family

$$\mathcal{F} = \left\{ \frac{1}{2mz} : m \in \mathbb{N} \right\}$$

on the open unit disk \mathbb{D} , and let $\phi(z) = 1/z$ and $\psi(z) \equiv 0$. Then clearly, for every $f, g \in \mathcal{F}$, f and g share $\phi(z)$, and $f^{(k)}$ and $g^{(k)}$ share $\psi(z)$ in \mathbb{D} . However, the family \mathcal{F} is not normal in \mathbb{D} . This shows that ϕ cannot be taken meromorphic in Theorem 1.6.

Further, for the same family \mathcal{F} , if we take $\phi(z) \equiv 0$ and $\psi(z) = 1/z^{k+1}$, then for every $f, g \in \mathcal{F}$, f and g share $\phi(z)$, and $f^{(k)}$ and $g^{(k)}$ share $\psi(z)$ in \mathbb{D} . But, the family \mathcal{F} is not normal in \mathbb{D} . This shows that the condition (3) in Theorem 1.6 is essential.

Example 1.9. Consider the family

$$\mathcal{F} = \left\{ z^{k+1} + \frac{1}{mz} : m \in \mathbb{N} \right\}$$

on the open unit disk \mathbb{D} , and let $\phi(z) = z^{k+1}$ and $\psi(z) = (k+1)!z$. Then clearly, for every $f, g \in \mathcal{F}$, f and g share $\phi(z)$, and $f^{(k)}$ and $g^{(k)}$ share $\psi(z)$ in \mathbb{D} . However, the family \mathcal{F} is not normal in \mathbb{D} . This shows that the condition $\phi^{(k)}(z) \not\equiv \psi(z)$ in Theorem 1.6 cannot be dropped.

Corollary 1.10. *Let $n \geq 2$ be a positive integer and $\psi(\not\equiv 0)$ be a function meromorphic on D . Let \mathcal{F} be a family of meromorphic functions in D such that for each pair of functions f and g in \mathcal{F} , f and g share the value 0, and $f^n f'$ and $g^n g'$ share the function ψ . Suppose further*

- (1) *for every common zero of f and ψ , the multiplicities m_f for f and m_ψ for ψ satisfy*

$$m_f \geq m_\psi + k + 1 \text{ for } k > 1, \text{ and}$$

$$m_f \geq 2m_\psi + 3 \text{ for } k = 1,$$

- (2) *for every $f \in \mathcal{F}$, f and ψ have no common poles in D .*

Then \mathcal{F} is normal on D .

Corollary 1.10 follows by setting $\mathcal{G} = \{f^{n+1}/(n+1) : f \in \mathcal{F}\}$ and applying Theorem 1.6 to this family with $\phi(z) \equiv 0$ and $k = 1$.

2. PROOF OF THE MAIN RESULT

For $z_0 \in \mathbb{C}$ and $r > 0$, we denote by $D_r(z_0)$ the open unit disk with centre z_0 and radius r , and $D'_r(z_0)$ the corresponding punctured disk. To prove our main result-Theorem 1.6, we require the following lemma:

Lemma 2.1. [2] *Let \mathcal{F} be a family of meromorphic functions in a domain D , all of whose zeros have multiplicity at least k . Then, if \mathcal{F} is not normal at z_0 , then for each $\beta : -1 < \beta < k$, there exist points $z_n \in D$ with $z_n \rightarrow z_0$, functions $f_n \in \mathcal{F}$ and positive numbers $\rho_n \rightarrow 0$ such that*

$$g_n(\zeta) := \rho_n^{-\beta} f_n(z_n + \rho_n \zeta)$$

converges locally uniformly with respect to the spherical metric in \mathbb{C} to a nonconstant meromorphic function g of finite order, all of whose zeros have multiplicity at least k .

Further, recall that the sets $\{E_\lambda\}_{\lambda \in \Lambda}$ are said to be *locally uniformly discrete* in D , if for each point $z_0 \in D$, there exists $\delta > 0$ such that either $E_\lambda \cap D_\delta(z_0)$ is an empty set or a singleton, one may refer to [1].

Proof of Theorem 1.6. Since normality is a local property, it is enough to show that \mathcal{F} is normal at each $z_0 \in D$. We distinguish the following cases.

Case I. Suppose that there exist $f \in \mathcal{F}$ such that $f(z_0) \neq \phi(z_0)$ and $f^{(k)}(z_0) \neq \psi(z_0)$. Then we can find $r > 0$ such that $D_r(z_0) \subset D$, and $f(z) \neq \phi(z)$ and $f^{(k)}(z) \neq \psi(z)$ in

$D_r(z_0)$ and so by the given sharing condition, $f \neq \phi$ and $f^{(k)} \neq \psi$ for every $f \in \mathcal{F}$ in $D_r(z_0)$. Now set $\alpha := \psi - \phi^{(k)}$ and consider the family

$$\mathcal{G} = \{g = f - \phi : f \in \mathcal{F}\}.$$

Then clearly $\alpha(z) \not\equiv 0$ and for every $g \in \mathcal{G}$, $g(z) \neq 0$ and $g^{(k)}(z) \neq \alpha(z)$. Thus by Theorem 1.3 \mathcal{G} is normal in $D_r(z_0)$. Since \mathcal{G} is normal if and only if \mathcal{F} is normal, \mathcal{F} is normal at z_0 .

Case II. Suppose that there exist $f \in \mathcal{F}$ such that $f(z_0) = \phi(z_0)$ or $f^{(k)}(z_0) = \psi(z_0)$. Then we can find $r > 0$ such that $D_r(z_0) \subset D$, and $f(z) \neq \phi(z)$ and $f^{(k)}(z) \neq \psi(z)$ in $D'_r(z_0)$ and so by the given sharing condition, $f(z) \neq \phi(z)$ and $f^{(k)}(z) \neq \psi(z)$ for every $f \in \mathcal{F}$ in $D'_r(z_0)$. Thus for any $z_1 \in D_r(z_0)$, there exists $\delta > 0$ such that every E_f has at most one point lying in $D_\delta(z_1)$, where

$$E_f := \{z \in D_r(z_0) : f(z) = \phi(z)\} \cup \{z \in D_r(z_0) : f^{(k)}(z) = \psi(z)\}.$$

Therefore, the sets $\{E_f\}_{f \in \mathcal{F}}$ are locally uniformly discrete in $D_r(z_0)$.

As in Case I, consider the family $\mathcal{G} = \{g = f - \phi : f \in \mathcal{F}\}$ and set $\alpha := \psi - \phi^{(k)}$. Then clearly the sets $\{E_g\}_{g \in \mathcal{G}}$ are locally uniformly discrete in $D_r(z_0)$, where

$$E_g = \{z \in D_r(z_0) : g(z) = 0\} \cup \{z \in D_r(z_0) : g^{(k)}(z) = \alpha(z)\}.$$

If $\alpha(z)$ is holomorphic in $D_r(z_0)$, then by [1, Theorem 4, p.49], \mathcal{G} is normal in $D_r(z_0)$ and hence \mathcal{F} is normal at z_0 . Suppose that $\alpha(z)$ is not holomorphic in $D_r(z_0)$. Assume that z_0 is a pole of $\alpha(z)$. Then we can find $\delta > 0$ such that $D_\delta(z_0) \subset D_r(z_0)$ and $\alpha(z)$ is holomorphic in $D'_\delta(z_0)$, and thus \mathcal{G} is normal in $D'_\delta(z_0)$. Next, consider the family

$$\mathcal{H} := \left\{ h(z) = \frac{g(z)}{\alpha(z)} : g \in \mathcal{G} \right\}.$$

Noting that z_0 is a pole of $\alpha(z)$ and for every $f \in \mathcal{F}$, f and ψ have no common poles implies that

- (a) for every $g \in \mathcal{G}$, g and α have no common poles and hence for every $h \in \mathcal{H}$, z_0 is a zero of h of multiplicity at least $k + 3$,
- (b) there exists $\eta > 0$ such that $D_\eta(z_0) \subset D_\delta(z_0)$ and for every $h \in \mathcal{H}$, $h \neq 1, \infty$ in $D_\eta(z_0)$.

We first prove that \mathcal{H} is normal at z_0 . Suppose on contrary that \mathcal{H} is not normal at z_0 . Then by Lemma 2.1, we can find a sequence $\{h_j\}$ in \mathcal{H} , a sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow 0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \rightarrow 0$ such that

$$H_j(\zeta) = h_j(z_j + \rho_j \zeta)$$

converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $H(\zeta)$ on \mathbb{C} , all of whose zeros have multiplicity at least $k + 3$. Also by Hurwitz theorem, we have $H \neq 1, \infty$ on \mathbb{C} . Thus by second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r, H) &\leq \overline{N}(r, H) + \overline{N}\left(r, \frac{1}{H}\right) + \overline{N}\left(r, \frac{1}{H-1}\right) + S(r, H) \\ &\leq \frac{1}{k+3} N\left(r, \frac{1}{H}\right) + S(r, H) \end{aligned}$$

$$\leq \frac{1}{k+3}T(r, H) + S(r, H),$$

which is a contradiction. Therefore \mathcal{H} is normal at z_0 . Now we turn to prove the normality of \mathcal{G} at z_0 .

Suppose that \mathcal{G} is not normal at z_0 . Since \mathcal{H} is normal at z_0 , it is equicontinuous at z_0 with respect to the spherical metric. Also $h(z_0) = 0$ for every $h \in \mathcal{H}$. Thus there exists $\delta_1 > 0$ such that $D_{\delta_1}(z_0) \subset D_\delta(z_0)$ and $|h(z)| \leq 1$ for every $h \in \mathcal{H}$ in $D_{\delta_1}(z_0)$. It follows that \mathcal{G} is a family of holomorphic functions in $D_{\delta_1}(z_0)$.

Let $\{g_n\}$ be a sequence in \mathcal{G} . Since \mathcal{G} is normal in $D'_{\delta_1}(z_0)$ but not at z_0 , there exists a subsequence of $\{g_n\}$, which we may take as $\{g_n\}$ itself, which converges locally uniformly on $D'_{\delta_1}(z_0)$ but not on $D_{\delta_1}(z_0)$. By the maximum modulus principle, we have $\{g_n\}$ converges locally uniformly to ∞ in $D'_{\delta_1}(z_0)$ and hence $\{h_n\}$ converges locally uniformly to ∞ in $D'_{\delta_1}(z_0)$, which is a contradiction to the fact that $|h(z)| \leq 1$ for every $h \in \mathcal{H}$ in $D_{\delta_1}(z_0)$. Thus \mathcal{G} is normal at z_0 and hence \mathcal{F} is normal at z_0 . ■

REFERENCES

1. J. Chang, *Normality of meromorphic functions and uniformly discrete exceptional sets*, Comput. Methods Funct. Theory **13** (2013), 47-63.
2. H.H. Chen and Y.X. Gu, *Improvement of Marty's criterion and its application*, Sci. China Ser. A **36**, 674-681 (1993).
3. M. Fang and L. Zalcman, *A note on normality and shared values*, J. Aust. Math. Soc. **76** (2004), 141-150.
4. M. Fang and L. Zalcman, *Normality and shared sets*, J. Aust. Math. Soc. **86** (2009), 339-354.
5. Y.X. Gu, *A normal criterion of meromorphic families*, Scientia Math. Issue I, 1979, 276-274.
6. W.K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
7. W.K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. Math. (2) **70** (1959), 9-42.
8. X. Liu and J. Chang, *A generalization of Gu's normality criterion*, Proc. Japan Acad. **88** Ser. A (2012).
9. S. Nevo, *On theorems of Yang and Schwick*, Complex Variable, Theory and Application, Vol. 46, issue 4, (2001), 315-321.
10. J.L. Schiff, *Normal Families*, Springer-verlag, New York, 1993.
11. W. Schwick, *Exceptional functions and normality*, Bull. London Math. Soc. **29** (1997), 425-432.
12. W. Schwick, *On hayman's alternative for families of meromorphic functions*, Complex variable, Theory and Application, vol. 32 (1997), 51-57.
13. Y. Xu, *On a result due to Yang and Schwick*, Sci. Sin. Math. **40**(5), (2010), 421-428.
14. L. Yang, *Normality of families of meromorphic functions*, Sci. Sinica A **9** (1986), 898-908.
15. P. Yang, X. Wang and X. Pang, *Normality criteria of meromorphic functions sharing a meromorphic function*, Journal of Mathematical Research with applications, Vol. 34, **5**, (2014), 543-548.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU-180 006, INDIA
E-mail address: kscharak7@rediffmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU-180 006, INDIA
E-mail address: virendersingh2323@gmail.com